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MAJOR SUBSETS OF α-RECURSIVELY ENUMERABLE SETS

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ABSTRACT

Let s2cf(α), s2p(α) and ts2p(α) denote the Σ_2-cofinality, Σ_2-projectum and the tame Σ_2-projectum of an admissible ordinal α. We show that if s2cf(α) < s2p(α), then no α-recursively enumerable set (α-r.e.) with complement of order type less than ts2p(α) can have a major subset. As a corollary, if s2cf(α) < s2p(α), then no hyperhypersimple α-r.e. set can have a major subset.

Let $\mathscr{E}(\alpha)$ denote the lattice of α-r.e. sets ordered by inclusion $\subseteq$. A set K is said to be α^*-finite if it is α-finite and of order type less than α^*, the Σ_1-projectum of α. If A and B are in $\mathscr{E}(\alpha)$, then we write $A =^* B$ if their symmetric difference is α^*-finite. Let $\mathscr{E}^*(\alpha)$ denote the lattice $\mathscr{E}(\alpha)$ modulo the α^*-finite sets. It has been found (Lerman [3], Chong and Lerman [1]) that the structure of $\mathscr{E}^*(\alpha)$ is more uniform for certain admissible ordinals $\alpha > \omega$. Thus if $\alpha = \aleph_1^L$, then there are no maximal α-r.e. sets nor hyperhypersimple α-r.e. sets, whereas if $\alpha = \aleph_\omega^L$, there are again no maximal α-r.e. sets while the class of hyperhypersimple α-r.e. sets, though nonempty, admits a neat characterization through order types of the complements of sets in $\mathscr{E}(\alpha)$. It is hoped that these nice properties will point a way towards establishing the decidability of the elementary theory of $\mathscr{E}(\alpha)$ (hence of $\mathscr{E}^*(\alpha)$, cf. Lerman [5]) and classifying those simple α-r.e. sets which form orbits (A and B in $\mathscr{E}^*(\alpha)$ are in the same orbit if there exists an automorphism Φ of $\mathscr{E}^*(\alpha)$ such that $\Phi(A) = B$).

The work on this paper was inspired by the attempt to study the relationship between major subsets of hyperhypersimple α-r.e. sets and their orbits. Recall that (cf. Lerman [4]) for A, B in $\mathscr{E}(\alpha)$, B is a major subset of A if

(1) $B \subset A$,

(2) $A - B$ is not α^*-finite, and

(3) for each C in $\mathscr{E}(\alpha)$, if $A \cup C = \alpha$, then $B \cup C =^* \alpha$.

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Leggett and Shore [2], generalizing an earlier result of Lerman [4], showed that if the Σ_2-cofinality (s2cf(α)) is not less than the Σ_2-projectum (s2p(α)) of α, then every α-r.e. non-α-recursive set has a major subset. It can be shown (cf. Leggett and Shore [2]) using this and the existence of hyperhypersimple α-r.e. sets for s2cf(α) < s2p(α), that for all admissible α, the simple α-r.e. sets do not have the same 1-type in the language of lattice theory. Two questions come naturally to mind:

(1) Can the Leggett–Shore result be generalized to all α, and

(2) What kind of role do major subsets play in the context of automorphisms of $\mathscr{E}^*(\alpha)$?

We show in this paper that the answer to (1) is in the negative. In fact, we show that if s2cf(α) < s2p(α), then no A in $\mathscr{E}(\alpha)$ with order type of $\alpha - A$ (ot($\bar{A}$)) less than ts2p(α), the tame Σ_2-projectum of α, has a major subset. On the other hand, Lerman, Shore and Soare recently showed [7, theorem 2.9] that there exist two atomless hyperhypersimple ω-r.e. sets which are not automorphic with respect to $\mathscr{E}^*(\omega)$. Now if $\alpha = \aleph_\omega^L$, for example, every hyperhypersimple α-r.e. set is atomless (this is an easy consequence of Chong–Lerman [1, corollary 4.13]). The proof of theorem 2.9 of [7] is done by exhibiting two atomless hyperhypersimple ω-r.e. sets, one with an r-maximal major subset and the other one without. This proof obviously cannot be lifted over to $\aleph_\omega^L$ since it follows from our result that major subsets do not exist for hyperhypersimple $\aleph_\omega^L$-r.e. sets. From this point of view, it seems that any two hyperhypersimple $\aleph_\omega^L$-r.e. sets might well be automorphic with respect to $\mathscr{E}^*(\aleph_\omega^L)$. We wish to thank the referee for pointing out a number of errors and inaccuracies in the original version of the paper. We also express our gratitude to M. Lerman for some very useful suggestions, and for discovering a gap in an earlier proof.

THEOREM. *Let* κ = s2cf(α) < s2p(α). *Let* A *be* α-r.e. *such that* ot($\bar{A}$) < ts2p(α). *Then* A *has no major subsets.*

We first state five lemmas whose proofs have appeared elsewhere. From now on, let κ = s2cf(α) < s2p(α) $\leqq$ ts2p(α).

LEMMA 1 (Lerman–Simpson [6, lemma 3.1]). *There is a simultaneously* α-r.e. *sequence of pairwise disjoint* α*-finite sets* $\{H_\eta \mid \eta < \kappa\}$ *such that*

(i) $\alpha = \bigcup \{H_\eta \mid \eta < \kappa\}$,

(ii) $\forall_{\eta<\kappa}$ $(\bigcup \{H_\xi \mid \xi < \eta\}$ *is* α*-finite*$)$.

LEMMA 2 (Chong–Lerman [1, lemma 2.4]). *Let* X *be an* α-r.e. *set such that* ot($\bar{X}$) < α^* *or* ts2p(α). *Then* X *is* α*-regular.*

LEMMA 3 (Lerman–Simpson [6, theorem 3.2]). *Let S be an unbounded tame Σ_2-subset of α. Let $K \subseteq \kappa$ be α-finite such that S is contained in $\bigcup\{H_\eta \mid \eta \in K\}$. Then there exist α-recursive sets X_0 and X_1 whose union is $\bigcup\{H_\eta \mid \eta \in K\}$ such that X_0 and X_1 split S into two parts, each unbounded in α.*

LEMMA 4 (Lerman [3, lemma 2.4]). *If X is an α-r.e. set of order type at least α^*, then X contains an α-r.e., non-α-recursive subset C.*

LEMMA 5 (Lerman [3, lemma 2.7]). *There is a nondecreasing α-recursive function t': $\alpha \times \kappa \to \alpha$ such that $\lim_{\sigma \to \alpha} t'(\sigma, \eta)$ exists for each $\eta < \kappa$ and is equal to $t(\eta)$. Furthermore, t: $\kappa \to \alpha$ is a strictly increasing Σ_2-cofinality function of α.*

Now let $B \subset A$ be α-r.e. such that $A - B$ is not α^*-finite. We will construct an α-r.e. set C such that $C \cup A = \alpha$ and $C \cup B \neq {}^*\alpha$. This will imply that B is not a major subset of A.

Let $\Gamma =$ the least x such that $(A - B)\restriction x$ is not α^*-finite.

If $\Gamma < \alpha$, then by Lemma 2, $A \restriction \Gamma$ and hence $\bar{A} \restriction \Gamma$ is α-finite. Let $C = \bar{A} \mid \Gamma \cup \{\nu \mid \Gamma \leqq \nu < \alpha\}$. Clearly $C \cup A = \alpha$ and $C \cap ((A - B)\restriction\Gamma)$ is empty so that $C \cup B \neq {}^*\alpha$ by the choice of Γ. C is furthermore α-r.e.

From now on, assume that $\Gamma = \alpha$.

Let the sets H_η and the function t be defined respectively as in Lemmas 1 and 5. Define $[\delta, \gamma) = \{\nu \mid \delta \leqq \nu < \gamma\}$. Let

$$\hat{K} = \{(\nu, \eta) \mid H_\eta \cap \bar{A} \neq \varnothing \quad \text{and} \quad H_\eta \cap (A - B) \cap [t(\nu), \alpha) \neq \varnothing\}.$$

Then $\hat{K}$ is a Σ_2-subset of κ so that it is α-finite.

For each $\nu < \kappa$, let $k(\nu)$ be the least (if exists) η such that (ν, η) is in $\hat{K}$.

Let ν_0 be the least $\nu < \kappa$ such that $k(\nu)$ is not defined. Suppose that ν_0 exists. Then for all $\eta < \kappa$, if

$$H_\eta \cap (A - B) \cap [t(\nu), \alpha) \neq \varnothing \qquad \text{for some } \nu \geqq \nu_0,$$

then $H_\eta \cap \bar{A} = \varnothing$.

Let $K' = \{\eta \mid \exists \nu \geqq \nu_0 (H_\eta \cap (A - B) \cap [t(\nu), \alpha) \neq \varnothing\}$. Again K' is α-finite. Define

$$C = \bigcup\{H_\eta \mid \eta \notin K'\}.$$

Then $C \supset \bar{A}$ and $C \cap H_\eta = \varnothing$ for all η in K', so that $C \not\supset {}^*\bar{B}$. This shows that B is not a major subset of A.

Now suppose that ν_0 does not exist so that $k(\nu)$ is defined for all $\nu < \kappa$. Let $K = \{k(\nu) \mid \nu < \kappa\}$.

For each $\eta = k(\nu)$ in K, let r_η be the least element of $H_\eta \cap (A - B) \cap [t(\nu), \alpha)$. We leave it to the reader to check using $\Gamma = \alpha$ that $R = \{r_\eta \mid \eta \in K\}$ is a non-α-recursive, tame Σ_2-set. For η in K, let κ_η be the order type of $H_\eta \cap (\bar{A} \cup \{r_\eta\})$. Let r_η be the ρ_ηth-element (in magnitude) of $H_\eta \cap (\bar{A} \cup \{r_\eta\})$. Clearly then there exists an $s < \mathrm{ts2p}(\alpha)$ such that $s > \kappa_\eta$ and ρ_η for all η in K.

LEMMA 6. *The following maps are α-finite*:

(a) $\eta \mapsto \kappa_\eta$,

(b) $\eta \mapsto \rho_\eta$.

PROOF. (a) Define $S(\eta, z)$ if and only if

$$\exists f\colon z \to H_\eta \text{ such that } f \text{ is strictly increasing}$$

$$\text{and } \exists\sigma\forall\tau \geqq \sigma\forall x < z\,(f(x) \in \bar{A}^\tau \quad \text{or} \quad f(x) = r_\eta).$$

By the fact that A is α-regular (by Lemma 2) and that R is tame Σ_2, we see that there exist z's such that $S(\eta, z)$. Also, if $S(\eta, z)$ then $S(\eta, z')$ for all $z' < z$. But S is a tame Σ_2-set bounded in $\mathrm{ts2p}(\alpha)$, hence it is α-finite. Then κ_η is just the least z such that $\sim S(\eta, z + 1)$.

(b) This time define $S(\eta, z + 1)$ if and only if

$$\exists f\colon z + 1 \to H_\eta \text{ such that } f \text{ is strictly increasing}$$

$$\text{and } \exists\sigma\forall\tau \geqq \sigma\forall x < z\,(f(x) \in \bar{A}^\tau \quad \text{and} \quad f(z) = r_\eta).$$

Now if $S(\eta, z + 1)$ then $z \leqq \rho_\eta$, and if $S(\eta, z + 1)$ then $S(\eta, z' + 1)$ for all $z' < z$. As in (a), S is α-finite. Then ρ_η is just the least z such that $\sim S(\eta, z + 2)$.

We now define by induction an α-recursive function T' which is an approximation of elements of $H_\eta \cap (\bar{A} \cup \{r_\eta\})$ at stage σ.

For η in K, let

$$T'(\sigma, \eta, \gamma) = \begin{cases} \text{the least } x \geqq \max\limits_{\delta < \gamma} \{T'(<\sigma, \eta, \gamma), T'(\sigma, \eta, \delta)\} \\ \text{such that } x \in \bar{A}^\sigma \cap H_\eta, \text{ if } \gamma < \rho_\eta < \kappa_\eta \text{ or } \rho_\eta < \gamma < \kappa_\eta; \\ \\ \text{the least } x \geqq \max\limits_{\delta < \gamma} \{T'(<\sigma, \eta, \gamma), T'(\sigma, \eta, \delta)\} \\ \text{such that } x \in H_\eta \cap [t'(\sigma, \nu), \sigma) - B^\sigma, \\ \text{where } k(\nu) = \eta \text{ and } \gamma = \rho_\eta. \end{cases}$$

The function t' used above is as defined in Lemma 5.

Thus $x = T'(\sigma, \eta, \gamma)$ if at stage σ, x appears to be the γth element of $H_\eta \cap (\bar{A} \cup \{r_\eta\})$. Our purpose is to select an α-r.e. set C containing $\bar{A}$ but disjoint from $R = \{r_\eta \mid \eta \in K\}$. Now whereas elements of $\bar{A} \cap H_\eta$ are enumerated in a Π_1 manner, the relation "$x = r_\eta$" is however Σ_2. Thus if $T'(\sigma, \eta, \gamma) =$ the γth element of $H_\eta \cap (\bar{A} \cup \{r_\eta\})$ (write this as $T(\eta, \gamma)$), and $\gamma > \rho_\eta$, it does not necessarily imply that $T'(\sigma, \eta, \rho_\eta) = r_\eta$. Note on the other hand that $T'(\sigma, \eta, \delta) \leqq T'(\tau, \eta, \gamma)$ for $\sigma \leqq \tau$ and $\delta \leqq \gamma$. Furthermore, it can be checked using the tame Σ_2-ness of R and the α-regularity of A (hence of $\bar{A}$) that $\lim_{\sigma \to \alpha} T'(\sigma, \eta, \gamma) = T(\eta, \gamma)$ exists for each $\gamma < \kappa_\eta$. Thus precautions will have to be taken to ensure that the α-r.e. set C to be constructed is disjoint from R. The following steps are designed precisely for such purposes.

For η in K, let

$$D_\eta^+ = \{\gamma \mid \rho_\eta < \gamma < \kappa_n \quad \text{and} \quad (\exists \sigma)(T'(\sigma, \eta, \gamma) = T(\eta, \rho_\eta) \quad \text{and}$$
$$(\exists \tau)(T'(\tau, \eta, \gamma) = T(\eta, \gamma) \quad \text{and} \quad T'(\tau, \eta, \rho_\eta) < T(\eta, \rho_\eta)\}.$$

Then D_η^+ consists of γ's which under T' approximate to the wrong value $T(\eta, \rho_\eta) = r_\eta$ at some stage and which arrive at the right values before ρ_η does.

Let

$$D_\eta = \{\gamma \mid \rho_\eta < \gamma < \kappa_\eta \quad \text{and} \quad (\exists \sigma)(T'(\sigma, \eta, \gamma) = T(\eta, \rho_\eta)\}.$$

Clearly $D_\eta^+ \subset D_\eta$.

LEMMA 7. *Each D_η, hence D_η^+, is a finite set.*

PROOF. Suppose that $\{\gamma_n\}_{n<\omega} \subset D_\eta$ is an infinite increasing sequence of ordinals. Let σ_n be the least σ such that $T'(\sigma, \eta, \gamma_n) = T(\eta, \rho_\eta)$. Then $\sigma_n < \sigma_m$ if $n > m$. Then $\{\sigma_n\}_{n<\omega}$ is an infinite descending sequence of ordinals, and this is a contradiction.

Next observe that $\eta \mapsto D_\eta^+$ and $\eta \mapsto D_\eta$ are α-finite maps. This can be seen (again) using the tame Σ_2-ness of R and the α-regularity of A. Each of D_η^+ and D_η is a finite subset of κ_η defined in a tame Σ_2-manner. Then $\bigcup = \{(\eta, \gamma) \mid \gamma \in D_\eta^+\}$ when ordered lexicographically has order type $\mathrm{ot}(K) \leqq \kappa$. The same is true when D_η^+ is replaced by D_η in $\bigcup$. Since $\kappa < \mathrm{ts2p}(\alpha)$, we see that U can be viewed as a tame Σ_2-set bounded below $\mathrm{ts2p}(\alpha)$, and hence α-finite. (Note: a tacit use here is made of the fact that $\mathrm{ot}(\bar{A}) < \mathrm{ts}2p(\alpha)$.)

It follows that there is an α-finite map $\hat{\gamma}$ such that for each $p \leqq \mathrm{card}(D_\eta^+)$, $\hat{\gamma}(\eta, p)$ is equal to the pth element of D_η^+. The same conclusion also holds if D_η^+ is replaced by D_η.

Now let γ be in D^+_η. For each σ, let $j(\sigma,\eta,\gamma)=$ the least stage τ such that $T'(\tau,\eta,\rho_\eta)\geqq T'(\sigma,\eta,\gamma)$. Let $j^{(0)}(\sigma,\eta,\gamma)=\sigma$. If $j^{(k)}(\sigma,\eta,\gamma)$ has been defined, let

$$j^{(k+1)}(\sigma,\eta,\gamma)=j(j^{(k)}(\sigma,\eta,\gamma),\eta,\gamma).$$

Then each $j^{(k)}$ is a partial α-recursive function and $j^{(k)}(\sigma,\eta,\gamma)$ is defined except when $T'(j^{(k-1)}(\sigma,\eta,\gamma),\eta,\gamma)>T(\eta,\rho_\eta)$.

Since γ is in D^+_η, there is a σ such that $T'(\sigma,\eta,\gamma)=T(\eta,\rho_\eta)$. Then $T'(j(\sigma,\eta,\gamma),\eta,\gamma)=T(\eta,\gamma)$. For each k, let σ_k, if defined, be the least stage σ such that

$$(*)\qquad T'(j^{(k)}(\sigma,\eta,\gamma),\eta,\gamma)=T(\eta,\gamma).$$

By our choice of σ_k and γ, clearly $T'(\sigma_k,\eta,\gamma)\leqq T(\eta,\rho_\eta)$. Notice that if $k>m$, then $\sigma_k<\sigma_m$ (when both are defined). There is then a largest k, denoted $k(\gamma)$, such that $(*)$ holds. Then if $\sigma<\sigma_{k(\gamma)}$, we have $T'(j^{(k)}(\sigma,\eta,\gamma),\eta,\gamma)<T(\eta,\gamma)$ for any k such that $j^{(k)}(\sigma,\eta,\gamma)$ is defined.

Lemma 8. *Let γ be in D^+_η. Let $k(\gamma)$ be as defined above. Then for all σ, $T'(j^{(k(\gamma))}(\sigma,\eta,\gamma),\eta,\gamma)\neq T(\eta,\rho_\eta)$ and there is a σ such that $T'(j^{(k(\gamma))}(\sigma,\eta,\gamma),\eta,\gamma)=T(\eta,\gamma)$.*

Proof. Suppose that there is a σ such that

$$T'(j^{(k(\gamma))}(\sigma,\eta,\gamma),\eta,\gamma)=T(\eta,\rho_\eta).$$

Then $T'(j^{(k(\gamma)+1}(\sigma,\eta,\gamma),\eta,\gamma)=T(\eta,\gamma)$. But this contradicts our choice of $k(\gamma)$. On the other hand, we clearly have $T'(j^{(k(\gamma))}(\sigma_{k(\gamma)},\eta,\gamma),\eta,\gamma)=T(\eta,\gamma)$.

Recall that there is an α-finite map $\hat\gamma$ such that $\hat\gamma(\eta,p)$ is the pth element of D^+_η. We leave it to the reader to check that $\hat\gamma(\eta,p)\mapsto k(\gamma(\eta,p))$ is an α-finite map. This α-finite map then provides us with an efficient method of picking out the $\hat\gamma(\eta,p)$th element of $H_\eta\cap(\bar A\cup\{r_\eta\})$, which is actually a member of $\bar A$, without "touching" r_η, according to Lemma 8.

Having taken care of D^+_η, let us consider $E_\eta=D_\eta-D^+_\eta$. As before, $n\mapsto E_\eta$ is α-finite. For $m\leqq\operatorname{card}(E_\eta)$, let $\varepsilon(\eta,m)$ denote the mth element of E_η. For convenience, let $r_\eta=\varepsilon(\eta,0)$. Then on E_η, the function T' satisfies the following:

(i) $T'(\sigma,\eta,\varepsilon)$ is defined for ε in E_η;

(ii) For all $\sigma\leqq\tau$, $1\leqq m\leqq\operatorname{card}(E_\eta)$, we have

$$T'(\sigma,\eta,\varepsilon(\eta,0))<T'(\sigma,\eta,\varepsilon(\eta,m))\quad\text{and}\quad T'(\sigma,\eta,\varepsilon(\eta,m))\leqq T'(\tau,\eta,\varepsilon(\eta,m));$$

(iii) $\lim_{\sigma\to\alpha}T'(\sigma,\eta,\varepsilon)=T(\eta,\varepsilon)$ exists for each ε in E_η;

(iv) If ε is in E_η and $T'(\sigma,\eta,\varepsilon)=T(\eta,\varepsilon)$, then $T'(\sigma,\eta,\rho_\eta)=T(\eta,\rho_\eta)$.

The following lemma evolved from Chong and Lerman [1, theorem 4.7].

LEMMA 9. *There is a partial α-recursive function $f' : \alpha \times K \times \omega \to \omega$ such that*
(a) *$f'(\sigma, \eta, m)$ is defined whenever $\varepsilon(\eta, m)$ is in E_η;*
(b) *for each $m \leqq \mathrm{card}(E_\eta)$, there is a σ such that for all $\tau \geqq \sigma$, all $n \leqq m$,*

$$f'(\tau, \eta, n) = f'(\sigma, \eta, n);$$

(c) *let $f(\eta, m) = \lim_{\sigma \to \alpha} f'(\sigma, \eta, m)$. Then the graph of f is α-finite.*
(d) *for any σ, if $f'(\sigma, \eta, m) = f(\eta, m)$, where $m \leqq \mathrm{card}(E_\eta)$, and $T'(\sigma, \eta, \varepsilon(\eta, m)) = x$, then $x \neq T(\eta, p_\eta) = r_\eta$.*

PROOF. Define f' by stages. Let $f'(\sigma, \eta, m)$ be undefined if $m = 0$ or if $m > \mathrm{card}(E_\eta)$. Now let η be in K and let $1 \leqq m \leqq \mathrm{card}(E_\eta)$. Compute $T'(\sigma, \eta, \varepsilon(\eta, m))$ and $T'(\sigma, \eta, \varepsilon(\eta, 0))$. Let τ_0 be the least $\lambda \leqq \sigma$ such that $T'(\lambda, \eta, \varepsilon(\eta, m)) = T'(\sigma, \eta, \varepsilon(\eta, m))$. If there does not exist a $\xi < \tau_0$ such that $T'(\xi, \eta, \varepsilon(\eta, m)) = T'(\tau_0, \eta, \varepsilon(\eta, 0))$, set $f'(\sigma, \eta, m) = 0$. Otherwise, let ζ be the least $\xi < \tau_0$ such that $T'(\xi, \eta, \varepsilon(\eta, m)) = T'(\tau_0, \eta, \varepsilon(\eta, 0))$. Let $f'(\sigma, \eta, m) = f'(\zeta, \eta, m) + 1$. Finally, if η is not in K, let $f'(\sigma, \eta, m)$ be undefined for all m and σ.

Observe that f' maps a subset of $\alpha \times K \times \omega$ into ω. Also, $f'(\sigma, \eta, m) = f'(\tau, \eta, m)$ if $T'(\sigma, \eta, \varepsilon(\eta, m)) = T'(\tau, \eta, \varepsilon(\eta, m))$. Since T' satisfies condition (iii) outlined above, and since each E_η is finite, we see that (a) and (b) are true. To prove (c), note that the graph of f is a Σ_2-subset of $K \times \omega \times \omega$. Since $K \subseteq \kappa$ and $\omega \leqq \kappa$, we see that graph $(f) \subseteq \kappa$ (since κ is an α-cardinal hence closed under pairing). But $\kappa < \mathrm{s2p}(\alpha)$, thus the graph of f is a Σ_2-set bounded below $\mathrm{s2p}(\alpha)$, hence α-finite, and this proves (c).

To prove (d), suppose that $f'(\sigma, \eta, m) = f(\eta, m)$ and let $T'(\sigma, \eta, \varepsilon(\eta, m)) = x$ (it is assumed here that $1 \leqq m \leqq \mathrm{card}(E_\eta)$). If (d) is false, then $x = T(\eta, \rho_\eta)$. Let ζ be the least ζ' such that $T'(\zeta', \eta, \varepsilon(\eta, m)) = T(\eta, \varepsilon(\eta, m))$. Then $\zeta > \sigma$. By condition (iv), $T'(\zeta, \eta, \rho_\eta) = T(\eta, \rho_\eta)$. But $T'(\zeta, \eta, \varepsilon(\eta, m)) = T(\eta, \varepsilon(\eta, m))$ and so $f'(\zeta, \eta, m) = f(\eta, m)$. Now let ξ be the least ξ' such that $\xi' \leqq \sigma$ and $T'(\xi', \eta, \varepsilon(\eta, m)) = T(\eta, \rho_\eta)$. Then $f'(\sigma, \eta, m) = f'(\xi, \eta, m)$ and $T'(\zeta, \eta, \rho_\eta) = T(\eta, \rho_\eta) = T'(\xi, \eta, \varepsilon(\eta, m))$ implies that $f'(\zeta, \eta, m) = f'(\xi, \eta, m) + 1$. We therefore get $f'(\sigma, \eta, m) \neq f(\eta, m)$, contrary to the choice of σ.

The proof of Lemma 9 is complete.

We are now ready to define an α-r.e. set C which will show that B is not a major subset of A.

$$C = \cup\{H_\eta \mid \eta \notin K\} \cup \{T'(\sigma, \eta, \gamma) \mid \sigma < \alpha, \eta \in K, \text{ and } \gamma \notin D_\eta\}$$
$$\cup \{T'(\sigma, \eta, \varepsilon(\eta, m)) \mid \sigma < \alpha, 1 \leqq m \leqq \operatorname{card}(E_\eta) \text{ and } f'(\sigma, \eta, m) = f(\eta, m)\}$$
$$\cup \{T'(j^{(k(\gamma(\eta,p)))}(\sigma, \eta, \hat{\gamma}(\eta, p)), \eta, \hat{\gamma}(\eta, p)) \mid$$
$$\sigma < \alpha \text{ and } \hat{\gamma}(\eta, p) \text{ is the } p\text{th element of } D_\eta^+\}.$$

Clearly $C \supset \bar{A}$. Using Lemmas 8 and 9 (d), we see that $C \cap R = \varnothing$, so that $C \not\supset {}^*B$. The proof of our Theorem is complete.

Corollary 10. *Let* s2cf(α) < s2p(α). *Let H be a hyperhypersimple α*-r.e. *set. Then H does not have a major subset.*

Proof. By Chong–Lerman [1, corollary 4.25] (also cf. Lerman [5, theorem 4.4]), $\operatorname{ot}(\bar{H}) < \alpha^*$ and $\bar{H}$ has a final segment of order type less than the tame Σ_2-projectum of α (ts2p(α)). Now by Lemma 2, H is α-regular. Let δ_0 be the least ordinal such that $[\delta_0, \alpha) \cap \bar{H}$ is of order-type less than ts2p(α). Let $H' = H \cup \delta_0$. Let B in $\mathscr{E}(\alpha)$ be a subset of H such that $H - B$ is not α^*-finite. Then $B \subset H'$ and $H' - B$ is not α^*-finite. By Theorem, there is a C in $\mathscr{E}(\alpha)$ which is a counterexample to B being a major subset of H'. Now if $\delta_0 = 0$, then $H = H'$ and so B is not a major subset of H. If $\delta_0 > 0$, then $h \restriction \delta_0$ is α-finite since H is α-regular. Let K be $H' - H$ which is just $(H' - H) \restriction \delta_0$. Let $C' = C \cup K$. Then we see that C' is a set showing that B is not a major subset of H.

Corollary 11. *Let* s2cf(α) < s2p(α). *If A in $\mathscr{E}(\alpha)$ is α-regular, and $\bar{A}$ has a final segment of order type less than* ts2p(α), *then A has no major subsets.*

Proof. Observe that in the proof of Corollary 10, the fact that $\bar{H}$ was of order type $< \alpha^*$ was used only to obtain the α-regularity of H.

References

1. C. T. Chong and M. Lerman, *Hyperhypersimple α-r.e. sets*, Ann. Math. Logic **9** (1976), 1–48.
2. A. Leggett and R. A. Shore, *Types of simple α-recursively enumerable sets*, J. Symbolic Logic **41** (1976), 681–694.
3. M. Lerman, *Maximal α-r.e. sets*, Trans. Amer. Math. Soc. **188** (1974), 341–386.
4. M. Lerman, *Types of simple α-recursively enumerable sets*, J. Symbolic Logic **41** (1976), 419–426.
5. M. Lerman, *Lattices of α-recursively enumerable sets*, to appear.

6. M. Lerman and S. G. Simpson, *Maximal sets in α-recursion theory*, Israel J. Math. **14** (1973), 236–247.

7. M. Lerman, R. A. Shore, and R. I. Soare, *R-maximal major subsets*, Israel J. Math. **31** (1978), 1–18.

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